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Renormalisation group approach to Delaunay percolation networks with toplogical disorder

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Abstract. A renormalisation group approach is developed for Delaunay percolating systems in two and three dimensions using a scaling transformation for a finite lattice in real space. Considering various renormalisation transformations for two- and three-dimensional Delaunay lattices, we determine the behaviour of the probabilities under a scale transformation and calculate the fixed point and connectedness length exponent. The fixed points for the two-dimensional bond lattice and the three-dimensional site lattice are 0.3229 and 0.1443 respectively, which are in excellent agreement with results of Monte Carlo simulations. The fixed point for the two-dimensional site lattice gives the value $\frac{1}{2}$ for the critical percolation probability which is equal to the known result for a fully triangulated lattice.

1. Introduction

Percolation has been actively studied using both momentum space and position space renormalisation group approaches. A variety of position space renormalisation group approaches have been developed recently and some of them are highly promising for the study of critical percolation phenomena.

In this paper we present a renormalisation group approach for calculating the critical behaviour of two- and three-dimensional Delaunay percolating systems using a scaling procedure in real space. For various regular lattices, the renormalisation study has thus far been carried out by several investigators who succeeded in obtaining precise estimates of the percolation threshold (Young and Stinchcombe 1975, Stinchcombe and Watson 1976, Reynolds *et al* 1977, Yuge and Murase 1978, Yuge 1978, 1979, Murase and Yuge 1979). There has, however, been no attempt to apply this approach to a Delaunay percolation system.

The basic method is analogous to the decimation procedure discussed by Kadanoff and Houghton (1975) for a spin system and applied to the critical behaviour for the percolation system by Young and Stinchcombe (1975). Our renormalisation method, which is based on the block formulation used in the derivation of the exact critical percolation probability for the triangular site lattice (Yuge 1978), is extended to Delaunay percolation networks in two and three dimensions and gives good results for the critical percolation probability p_c and for the connectedness length exponent ν.

In the following section we review a technique of constructing a two-dimensional Voronoi tessellation, easily extendable to three dimensions. In § 3, we present some

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principles of the real space renormalisation group approach to percolation and demonstrate an example of the bond problem on the Delaunay percolation network in two dimensions. Moreover, we present the renormalisation technique for the site problem of the Delaunay percolation network in two dimensions and the results obtained by the renormalisation group approach. We find that our results for the fixed point of the site problem agree with the conjecture of Sykes and Essam (1964) for the percolation threshold of the fully triangulated site lattices. We also present in § 4 the renormalisation technique in three dimensions and the results for the fixed point and the connectedness length exponent for the Delaunay site network.

2. The Voronoi tessellations and Delaunay network

The Voronoi tessellation, also known as the Wigner–Seitz cell construction, is a method for subdividing a continuum into random convex polytopes, i.e. polygons in two dimensions and polyhedra in three dimensions. The tessellation is carried out by first distributing particle points (for example 'Poisson points') randomly in space, and then drawing the perpendicular bisector of each line segment connecting each pair of points. Next we construct the minimum polytope about the point formed by the intersection of the bisectors. The Delaunay network, which is the dual of the Voronoi tessellation, is constructed by representing each polygon as a site, so that the line segments between neighbouring points become the bonds. For randomly distributed particle points, Delaunay networks are almost surely simplicial complexes, which consist of triangles(2D) or tetrahedra(3D) alone. An illustration of the two-dimensional example is given in figure 1 where the Delaunay triangles are shown by full lines and the corresponding Voronoi polygons are shown by broken lines. The coordination number z of a site in the network is equal to the number of edges of the polygons that surround the site.

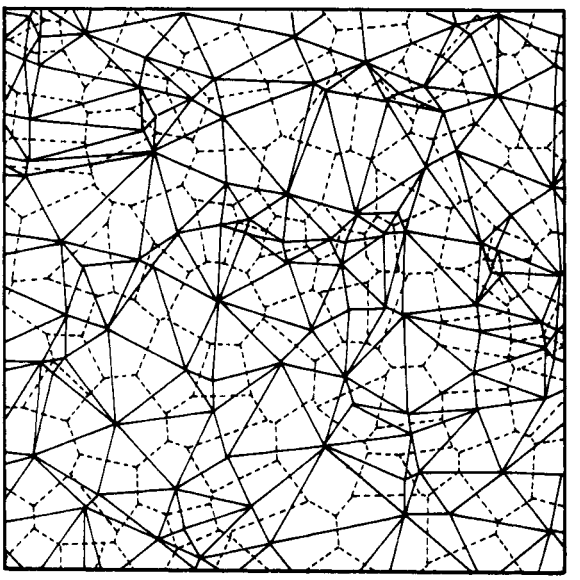

Figure 1. Voronoi polygons (broken line) and Delaunay triangles (full line) in two dimensions.

The percolation properties can be defined by the procedure of removing conducting elements (sites or bonds) at random from an infinitely large tessellation of conductors. In the network, a cluster is defined as an isolated finite group of conducting elements in electrical contact with one another. If we remove so many elements that the fraction p of conducting elements is below the percolation threshold p_c, then the system behaves as an insulator. If the fraction p is above the percolation threshold, a continuous conducting path across the system exists for any infinite system. The examples to be considered here are the site and bond percolation networks in two dimensions (see Winterfeld *et al* 1981, Jerauld *et al* 1984a) and the site percolation network in three dimensions (see Jerauld *et al* 1984b), in which an element (site or bond) of the Delaunay network is present (conducting) with probability p and absent (insulating) with probability $1-p$.

3. The renormalisation group approach to two-dimensional Delaunay networks

We start by partitioning a lattice into cells which cover the lattice and preserve its topological structure. Our renormalisation procedure will be worked out for the bond problem and the site problem. We discuss the renormalisation transformations for the two problems separately.

3.1. The bond problem

We shall consider the simplest example in which twelve bonds on an original lattice scale into three bonds on a new lattice (figure 2). An illustration of the basic scaling procedure is provided in figures 2–5. The renormalisation procedure for the Delaunay bond network is carried out by first choosing any one polygon and its nearest-neighbour polygons. If the number of nearest-neighbour polygons are not sufficient, we choose next-nearest-neighbour polygons that are in contact with the two nearest-neighbour polygons. In the Voronoi tesselation, a twelve-bond cell consists of a group of nine

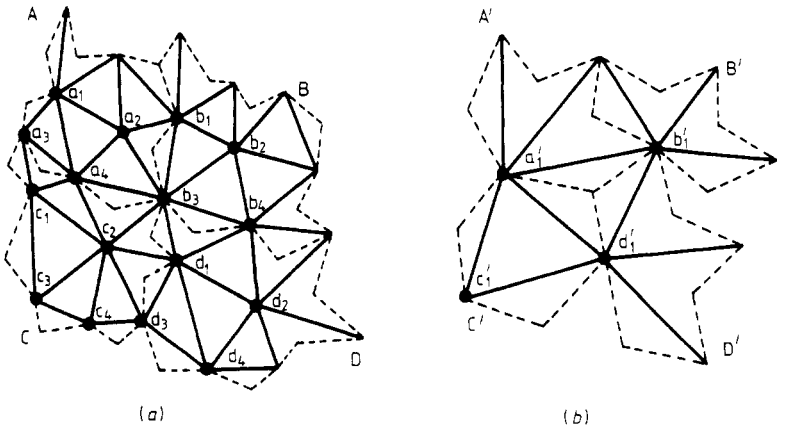

Figure 2. Rescaling a lattice by forming cells out of groups of bonds. (*a*) Twelve-bond cells with four sites on the original Delaunay lattice enclosed by the broken line. (*b*) Three-bond cells with one site on the new lattice enclosed by the broken line. In this example, the cell A on the original lattice scales into the cell A′ on the new lattice, and also cells B, C, and D scale into cells B′, C′, and D′, respectively.

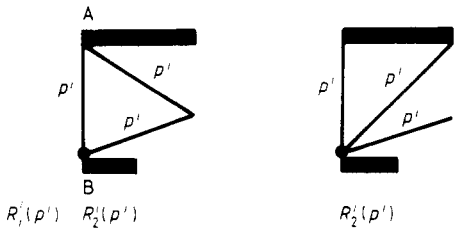

Figure 3. All the topologically distinct Delaunay graphs for a three-bond cell on the new lattice. Three-bond cells are sandwiched between two electrodes A and B.

neighbouring polygons. Correspondingly, in the Delaunay network, a twelve-bond cell is a group of four sites connected by twelve bonds as shown in figure 2(a). Under this transformation, moreover, we choose the twelve-bond cell so that it can be fitted onto a frame of a square lattice by distortion. Next, the new lattice is obtained by removing three sites at random from the four sites in the cell and reconstructing the Delaunay network from the remaining sites (figure 2(b)). A three-bond cell on the new lattice is a group of one site and three bonds. There are only two topologically distinct graphs for a three-bond cell as shown in figure 3.

Thus, we have replaced the twelve bonds in each cell on the original lattice by the three bonds on the new lattice. Next we consider all topologically distinct graphs of the cell consisting of twelve bonds. The graphs of the twelve-bond cell shown in figure 4 are all topologically distinct graphs for that cell. There are some topologically distinct graphs with coordination number $z = 5$–7 associate with site C. For other coordination numbers, there is only one graph. Such graphs can be fitted onto a frame of the square lattice by distortion. These distorted graphs are sandwiched between two electrodes, as illustrated on the right of the original graph shown in figure 4. The bonds in the original graph are independently present (conducting) with probability p and absent (insulating) with probability $1-p$. When an electric voltage is applied to the distorted graph sandwiched between two electrodes A and B in figure 5, the graph being conductive is defined as a graph in which a continuous conducting path from electrode A to electrode B exists. To the graph i with coordination number z in figure 4 are given the weighting factor $N_i^{(z)}$ (the frequency of the graph i in graphs with the coordination number z), the probability $R_i^{(z)}(p)$ of the graph being conductive, and its coordination number z associated with the site C. The coordination number z in this transformation ranges from $z = 4$ to $z = 8$. The probability of finding sites with coordination number between 4 and 8 in the network is about 0.95, which can be estimated from the distribution of coordination numbers obtained by Monte Carlo simulations (Winterfeld *et al* 1981, Jerauld *et al* 1984a).

Some of the configurations for one graph that arise in the position space renormalisation transformation using a twelve-bond cell are shown in figure 5. In those diagrams, conducting bonds are represented with a full line; broken lines represent insulating bonds.

The sum of the probability p_j of those configurations is equal to the probability $R(p)^{(z)}_{\text{graph}}$ of a graph being conductive, i.e.

$$R(p)^{(z)}_{\text{graph}} = \sum_j p_j = p^{11} + 11p^{10}(1-p) + 55p^9(1-p)^2 + 165p^8(1-p)^3 + 326p^7(1-p)^4 + 432p^6(1-p)^5 + 372p^8(1-p)^6 + 188p^5(1-p)^7 + 49p^3(1-p)^8 + 5p^2(1-p)^9. \tag{1}$$

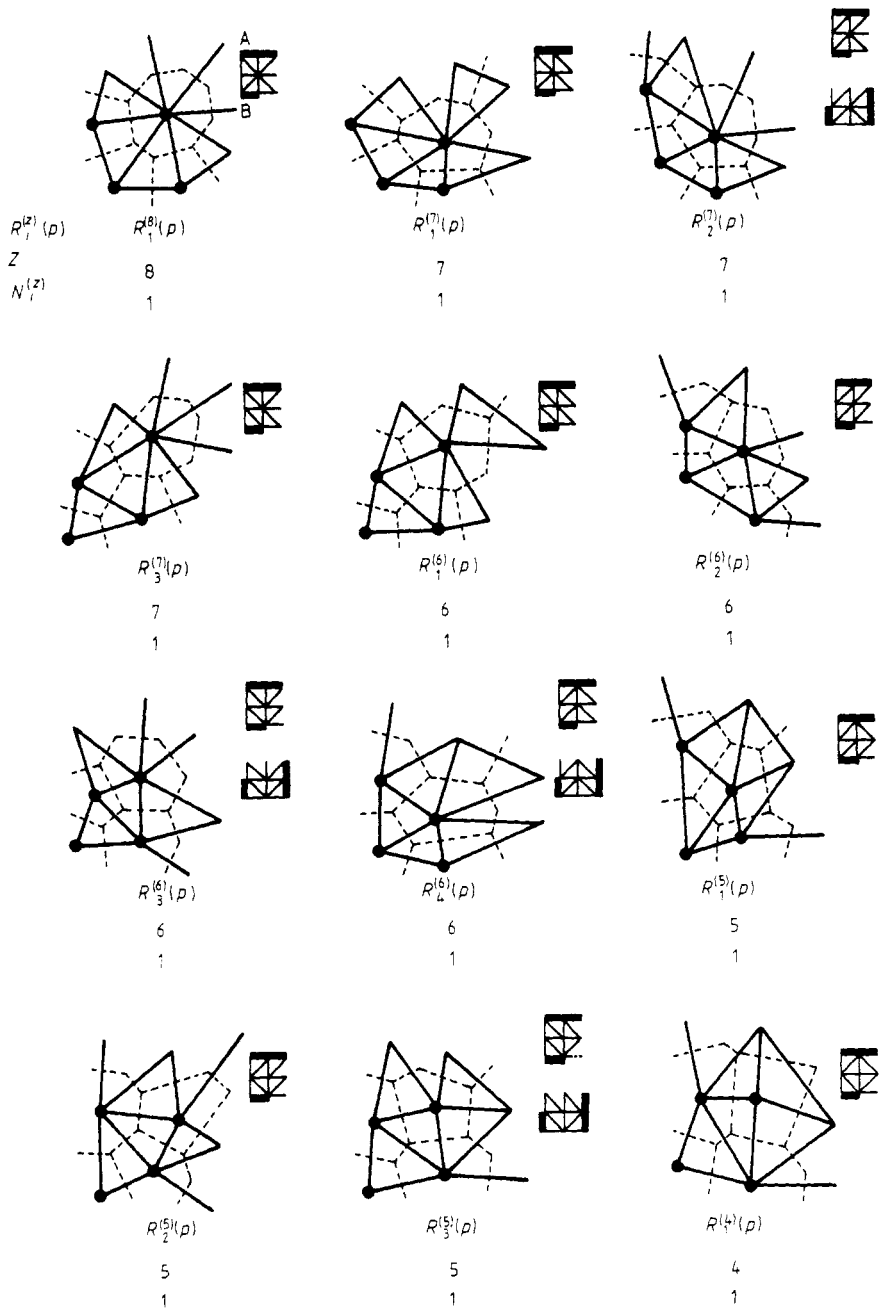

Figure 4. All the topologically distinct Delaunay graphs (full line) and Voronoi polygons for a twelve-bond cell are shown together with their weighting factor $N_i^{(z)}$, the probability $R_i^{(z)}(p)$ of a graph being conductive and its coordination number z associated with the site C. The distorted structure illustrated on the right is topologically equivalent to the original graph.

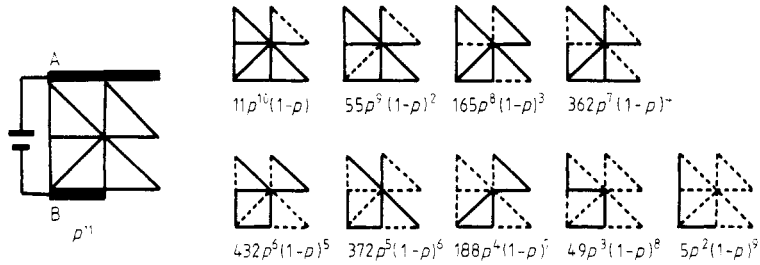

Figure 5. Some of the configurations of paths constructed between two electrodes A and B using a twelve-bond cell. Each bond shown as a full line and broken line represent bonds occupied with probability p and $1-p$, respectively.

The probability $R'(p')^{(z)}_{\text{graph}}$ of a graph on the new lattice is also expressed as

$$R'(p')^{(z)}_{\text{graph}} = \Sigma_j p'_j = p'^3 + 3p'^2(1-p') + p'(1-p')^2. \tag{2}$$

Similarly, we can obtain the probability $R'_i(p')$ and $R_i^{(z)}(p)$ of graphs in figures 3 and 4, considering the paths which contribute to the conductance of the cell (see appendix 1). There is only one topologically distinct graph except for the coordination number $z = 5$–7 associated with site C. For the coordination number $z = 5$–7, there are some topologically distinct graphs which, we assume, occur with the same probability. When these graphs are not statistically isotropic, i.e. when the probability $R_i^{(z)}(p)_x$ that the graph is horizontally conductive is inequal to the probability $R_i^{(z)}(p)_y$ of its being vertically conductive, the probability $R_i^{(z)}(p)$ of a given graph is defined as their arithmetic mean. In general, the probability $R_i^{(z)}(p)$ that the graph is conductive is defined as the arithmetic mean of the probabilities of its being conductive vertically and horizontally.

The twelve-bond cell scales into the three-bond cell; given that the bonds are independently occupied with probability p, then the occupation probability p' for the bonds of the new lattice may be derived from the renormalisation transformation

$$\mathcal{R}'\{R'(p')\} = \mathcal{R}\{R(p)\} \tag{3}$$

where

$$\mathcal{R}\{R(p)\} = \sum_z \left\{ \sum_{\text{graph}} R(p)^{(z)}_{\text{graph}} P^{(z)}_{\text{graph}} \right\} \Phi(z) \tag{4}$$

and

$$\mathcal{R}'\{R'(p')\} = \sum_z \left\{ \sum_{\text{graph}} R'(p')^{(z)}_{\text{graph}} P'^{(z)}_{\text{graph}} \right\} \Phi(z). \tag{5}$$

Here $\mathcal{R}\{R(p)\}$ and $\mathcal{R}'\{R'(p')\}$ are the configurational averages for the probabilities that these graphs on the original and the new lattices are conductive, respectively; z is the coordination number associated with the site C of these graphs and $\Phi(z)$ is the distribution of coordination numbers. The summation Σ_z is taken over all states of coordination number z and the summation Σ_{graph} is over all topologically distinct graphs of the coordination number z. Furthermore, $P^{(z)}_{\text{graph}}$ and $P'^{(z)}_{\text{graph}}$ are concerned with the probability that the graph is found in graphs with the coordination number z.

The configurational average $\mathcal{R}$ with which the 'system' (all topologically distinct graphs) becomes conductive is obtained by summing the probabilities $R_i^{(z)}(p)$ of these

graphs:

$$\mathcal{R}\{R(p)\} = \sum_z \left\{ \sum_{\text{graph}} R(p)^{(z)}_{\text{graph}} P^{(z)}_{\text{graph}} \right\} \Phi(z)$$

$$= \sum_z \left\{ \sum_{\text{graph}} R(p)^{(z)}_{\text{graph}} N^{(z)}_{\text{graph}} \left(\sum_{\text{graph}} N^{(z)}_{\text{graph}} \right)^{-1} \right\} \Phi(z)$$

$$\simeq \sum_{z=4}^{8} \left\{ \sum_i R_i^{(z)}(p) N_i^{(z)} \left(\sum_i N_i^{(z)} \right) \right\}^{-1} \Phi(z) \left(\sum_{z=4}^{8} \Phi(z) \right)^{-1}. \tag{6}$$

The distribution of coordination numbers can be estimated from Monte Carlo simulations (Winterfeld *et al* 1981, Jerauld *et al* 1984a). On the other hand, there are only two topologically distinct graphs for a three-bond cell on the new lattice and these graphs are shown in figure 3. The configurational average $\mathcal{R}'$ of the system being conductive on the new lattice is given by

$$R'\{R'(p')\} = \sum_{i=1}^{2} R_i'(p')/2 = (3p' - p'^3)/2. \tag{7}$$

Equation (3) serves as a simple though highly approximate position space renormalisation group transformation with fixed points given by

$$\mathcal{R}'\{R'(p^*)\} = \mathcal{R}\{R(p^*)\}. \tag{8}$$

From equations (6)-(8), we find two trivial fixed points at $p^* = 0, 1$ and also a non-trivial fixed point at

$$p^* = 0.3213. \tag{9}$$

A non-trivial fixed point gives the critical percolation probability p_c for the finite cell. This result for p^* is in excellent agreement with the result 0.321 ± 0.003 computed by the cluster moment of Dean (1963). The percolation threshold of the Delaunay bond network was recently determined to be $p_c = 0.3287 \pm 0.016$ for a network of 10 000 sites (Winterfeld *et al* 1984a).

On the other hand, since our transformation rescales the lattice spacing by a factor of b, which is the change of the linear density of sites in the network, the average distance between any pair of points is smaller by a factor of b. Near the fixed point, we may linearise the renormalisation group equation (3) to obtain

$$\left. \frac{\mathrm{d}\mathcal{R}'\{R'(p')\}}{\mathrm{d}p'} \right|_{p'=p^*} (p' - p^*) = \left. \frac{\mathrm{d}\mathcal{R}\{R(p)\}}{\mathrm{d}p} \right|_{p=p^*} (p - p^*) \tag{10}$$

where

$$\lambda = \left. \frac{\mathrm{d}\mathcal{R}\{R(p)\}}{\mathrm{d}p} \right|_{p=p^*} \left(\left. \frac{\mathrm{d}\mathcal{R}'\{R'(p')\}}{\mathrm{d}p'} \right|_{p'=p^*} \right)^{-1}. \tag{11}$$

The connectedness length exponent is given by

$$\nu = \ln b / \ln \lambda. \tag{12}$$

For the simplest example $b = 2$, from the scaling transformations (6) and (7) the eigenvalue and the connectedness length exponent are given by

$$\lambda = 1.5930 \qquad \nu = 1.4888. \tag{13}$$

The possibly exact value of $\nu = \frac{4}{3}$ (den Nijs 1979) is in fairly good agreement with this result in view of the small size of the cell.

3.2. The site problem

Similar calculations may be carried out for the site problem using the corresponding renormalisation transformations.

3.2.1. Four-site to one-site transformation. We shall consider the simplest example in the site problem in which four sites on the original lattice scale into one site on a new lattice. An illustration of the basic scaling procedure on this lattice is provided in figures 6 and 7. In the Voronoi tessellation, the renormalisation procedure for the Delaunay site network is carried out by choosing four neighbouring polygons. Correspondingly, in the network, we choose a four-site cell so that it can be fitted onto a frame of a square lattice by distortion. The four-site cell consistes of a group of four sites connected by five bonds as shown in figure 6(a). Next, the new lattice is obtained by removing three sites at random from the four sites in the cell and reconstructing the Delaunay site network from the remaining sites (figure 6(b)).

Thus, we have replaced the four sites in each cell on the original lattice by one site on the new lattice. Next we consider all topologically distinct graphs consisting of four sites. For the four-site cell, there is only one topologically distinct graph shown in figure 7. This graph is distorted and sandwiched between two electrodes as shown in the figure.

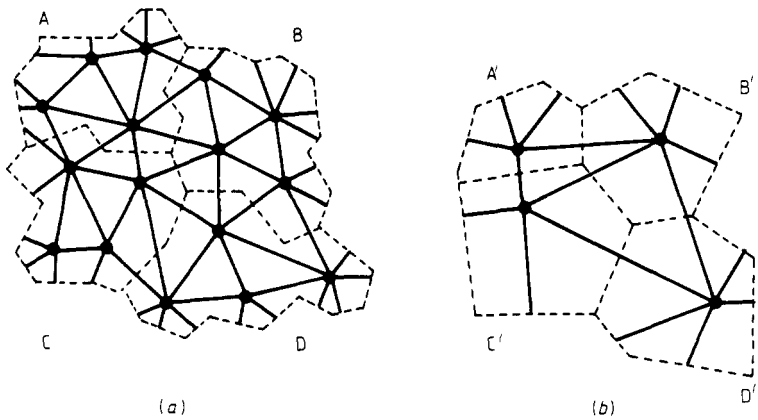

Figure 6. Rescaling a lattice by forming cells out of groups of sites. (a) Four-site cells on the original Delaunay lattice (enclosed by the broken line) are shown with dots representing occupied sites with probability p. (b) One-site cells on the new lattice (enclosed by the broken line) are shown with dots representing occupied with probability p'. In this example, the cell A on the original lattice scales into the cell A′ on the new lattice, and also cells B, C and D scale into cells B′, C′ and D′, respectively.

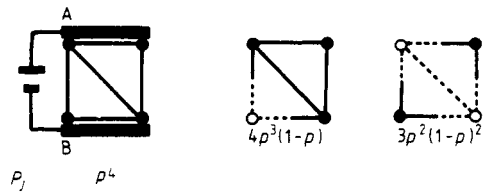

Figure 7. Some of the configurations of paths conducting between two electrodes A and B. Four-site cell on the original lattice; each black site and each white site represent sites occupied with probability p and $1-p$, respectively.

The average probability $\mathcal{R}$ with which the system becomes conductive is obtained, taking account of paths which contribute to the conductance of the graph:

$$\mathcal{R}\{R(p)\} = \sum_j p_j = p^4 + 4p^3(1-p) + 3p^2(1-p)^2. \tag{14}$$

On the other hand, for the one-site cell with probability p', the probability $\mathcal{R}'$ equals p', so that

$$\mathcal{R}'\{R'(p')\} = p'. \tag{15}$$

From equations (3), (9), (14) and (15), the fixed point value of p is found to be

$$p^* = \tfrac{1}{2}. \tag{16}$$

This result for p^* is in complete agreement with conjecture of Sykes and Essam (1964) for the percolation probability of a fully triangulated site lattice. The transformation has an eigenvalue $\lambda = 1.5$. Thus the connectedness length exponent ν is given by

$$\nu = 1.7095. \tag{17}$$

This renormalisation procedure of 'four-site to one-site transformation' is equivalent to that of the triangular site lattice (Yuge 1978).

3.2.2. Nine-site to one-site transformation. As in the bond problem, this renormalisation procedure for the Delaunay site network is carried out by first choosing any one polygon and its nearest-neighbour polygons. If the number of such nearest-neighbour polygons are not sufficient, we choose next nearest-neighbour polygons that are in contact with the two nearest-neighbour polygons. In the network, a nine-site cell is a group of nine sites connected by bonds. Under this transformation, moreover, we choose a nine-site cell so that it can be fitted onto a frame of a square lattice by distortion. Next, the new lattice is obtained by removing eight sites at random from the nine sites in the cell and reconstructing the Delaunay network from the remaining sites.

Thus, we have replaced the nine sites in each cell on a original lattice by the one site on the new lattice. Next, we consider all topologically distinct graphs of the cell consisting of nine sites. The graphs of the nine-site cell shown in figure 8 are all topologically distinct graphs for that cell. There are two topologically distinct graphs with coordination number $z = 6$ associated with the central site C. For other coordination numbers, there is only one graph. Such graphs can be fitted onto the frame of the square lattice by distortion. These distorted graphs are sandwiched between two electrodes, as illustrated on the right of the original graph in figure 8. Sites in the original graph are independently present (conducting) with probability p. To the graph i with coordination number z in figure 8 are given the weighing factor $N_i^{(z)}$ (the frequency of the graph i in graphs with the coordination number z), the probability $R_i^{(z)}(p)$ of the graph being conductive and its coordination number z associated with the central site C. The coordination number z in this transformation ranges from $z = 4$ to $z = 8$. We can obtain the probability $R_i^{(z)}(p)$ of each graph in figure 8 considering the paths which contribute to the conductance of the graph (see appendix 2). There is only one topologically distinct graph except for the coordination number $z = 6$. For the coordination number $z = 6$, there are two topologically distinct graphs which, we assume, occur with the same probability. The probability $R_i^{(z)}(p)$ that the graph is conductive is defined as the arithmetic mean of the probabilities of its being conductive vertically and horizontally.

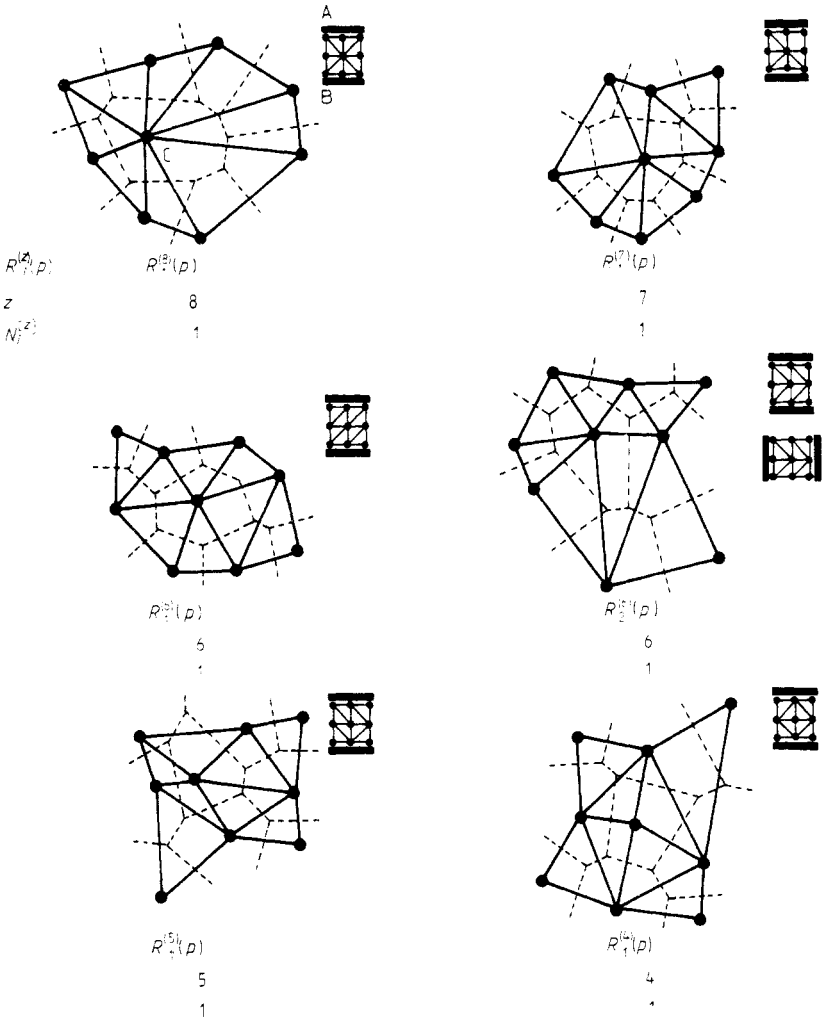

Figure 8. All the topologically distinct Delaunay graphs (full line) and the Voronoi polygons (broken line) for a nine-site cell are shown together with their weighting factor $N_i^{(z)}$, the probability $R_i^{(z)}$ and its coordination number z. The distorted structure illustrated on the right is topologically equivalent to the original graph.

The average probability $\mathcal{R}$ with which the system becomes conductive is obtained by summing the probabilities $R_i^{(z)}(p)$ of these graphs:

$$\mathcal{R}\{R(p)\} = \sum_{z=4}^{8} \left\{ \sum_i R_i^{(z)}(p) N_i^{(z)} \left(\sum_i N_i^{(z)} \right)^{-1} \right\} \Phi(z) \left(\sum_{z=4}^{8} \Phi(z) \right)^{-1}. \tag{18}$$

On the other hand, for the one-site cell with probability p', the probability $\mathcal{R}'$ equals p', so that

$$\mathcal{R}'\{R'(p')\} = p'. \tag{19}$$

From equations (3), (9), (18) and (19), the fixed point value of p is found to be

$$p^* = \tfrac{1}{2}. \tag{20}$$

This result for p^* also is in complete agreement with the critical percolation probability of a fully triangulated site lattice. The transformation has an eigenvalue $\lambda = 1.9235$. Thus the connectedness length exponent ν is given by

$$\nu = 1.6794. \tag{21}$$

3.2.3. Nine-site to four-site transformation. We shall consider a more intricate example in which nine sites on the original lattice scale into four sites on a new lattice. Similar to the nine-site to one-site transformation, this renormalisation procedure is carried out by first choosing any one polygon and its nearest-neighbour polygons. In the Delaunay network, a nine-site cell is a group of nine sites connected by bonds. Under this tranformation, moreover, we choose a nine-site cell so that it can be fitted onto a frame of a square lattice by distortion. Next, the new lattice is obtained by removing five sites at random from the nine sites in the cell and reconstructing the Delaunay network from the remaining sites. A four-site cell on the new lattice is a group of four sites connected by five bonds. There is only one topologically distinct four-site cell, which is equivalent to the graph shown in figure 7.

Thus, we have replaced the nine sites in each cell on the original lattice by the four sites on the new lattice. Next we consider all topologically distinct graphs of the cell consisting of nine sites. Graphs of the nine-site cell which are all topologically distinct are equivalent to those shown in figure 8.

The average probability $\mathcal{R}$ with which the system becomes conductive is obtained by summing the probabilities $R_i^{(z)}(p)$ of these graphs:

$$\mathcal{R}\{R(p)\} = \sum_{z=4}^{8} \left\{ \sum_i R_i^{(z)}(p) N_i^{(z)} \left(\sum_i N_i^{(z)} \right)^{-1} \right\} \Phi(z) \left(\sum_{z=4}^{8} \Phi(z) \right)^{-1}. \tag{22}$$

On the othe hand, the probability $\mathcal{R}'$ of the system being conductive on the new lattice is given by

$$\mathcal{R}'\{R'(p')\} = R'(p') = 3p'^2 - 2p'^3. \tag{23}$$

From equations (3), (9), (22) and (23), the fixed point value of p is found to be

$$p^* = \tfrac{1}{2}. \tag{24}$$

This result for p^* is again in complete agreement with the critical percolation probability of a fully triangulated site lattice. The transformation has an eigenvalue $\lambda = 1.9235$. Thus the connectedness length exponent ν is given by

$$\nu = 1.6303. \tag{25}$$

4. The three-dimensional Delaunay network

We extend the previous procedure to the Delaunay percolation network in three dimensions. Our simple scaling procedure is defined by a renormalisation transformation on a finite lattice sandwiched between two electrodes. We shall consider the simplest example where the eight-site cell on the original lattice scales into the one-site cell on a new lattice. In the same manner as in the two dimensional tessellation, the renormalisation procedure for the Delaunay site network in three dimensions is carried out by first choosing eight neighbouring polyhedra. Correspondingly, in the network, an eight-site cell consists of a group of four sites connected by nineteen bonds. Under

this transformation, moreover, we choose the eight-site cell so that it can be fitted onto a frame of a cubic lattice by distortion. Next, the new lattice is obtained by removing seven sites at random from the eight sites in the cell and reconstructing the Delaunay network from the remaining sites.

Thus, we have replaced the eight sites in each cell on the original lattice by the one site on the new lattice. Next, we consider all topologically distinct graphs of the cell consisting of eight sites. The graphs of the eight-site cell shown in figure 9 are all topologically distinct graphs for that cell which, we assume, occur with the same probability in the tessellation. These graphs can be fitted onto the frame of the cubic lattice by distortion and subdivided by some tetrahedra. The sites in the original graph are independently present (conducting) with probability p. To each graph i of figure 9 are given the weighting factor N_i (the frequency of the graph i) and the probability $R_i(p)$ of the graph being conductive.

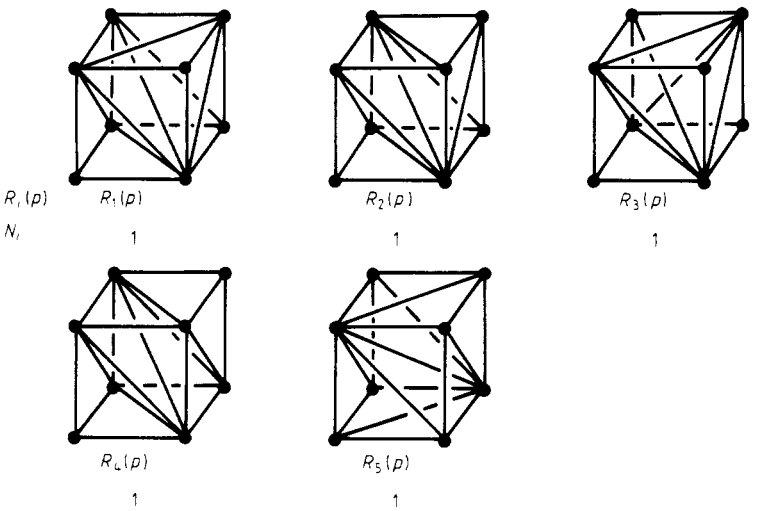

Figure 9. All the topologically distinct graphs are shown together with their weighting factor N_i and the probabilitty $R_i(p)$ of the graph being conductive.

We can obtain the probability $R_i(p)$ of each graph in figure 9, considering the paths which contribute to the conductance of the graph. Configurations of one graph that arise in the renormalisation transformation using an eight-site cell is shown in figure 10. In those diagrams, each conducting site has been connected with full lines; the white sites connected with broken lines represent insulating sites. The sum of the probability p_j of those configurations is equal to the probability of a graph being conductive vertically, i.e.

$$R_1(p)_z = \sum_j p_j = p^8 + 8p^7(1-p) + 28p^6(1-p)^2 + 56p^5(1-p)^3 + 65p^4(1-p)^4 + 39p^3(1-p)^5 + 9p^2(1-p)^6. \tag{26}$$

When the probability $R_i(p)_z$ is the probability that the graph is vertically conductive and the probabilities $R_i(p)_x$ and $R_i(p)_y$ are the probabilities of its being conductive for the two horizontal directions, the probability $R_i(p)$ of a given graph is defined as their arithmetic mean (see appendix 3).

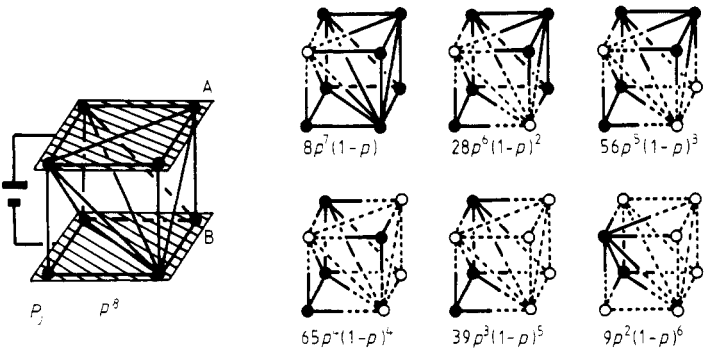

Figure 10. Some of the configurations of paths conducting between two electrodes A and B using an eight-site cell in three dimensions. Each black site and each white site represent sites occupied with probability p and $1-p$, respectively.

The average probability $\mathscr{R}$ with which the system becomes conductive is obtained by summing the probabilities $R_i(p)$ of these graphs:

$$\mathscr{R}\{R(p)\} = \sum_{i=1}^{5} R_i(p) N_i \left(\sum_{i=1}^{5} N_i \right)^{-1}. \tag{27}$$

For the one-site cell with probability p', the probability $\mathscr{R}'$ equals p', so that

$$\mathscr{R}'\{R'(p')\} = p'. \tag{28}$$

From equations (3), (9), (27) and (28), the fixed point value of p is found to be

$$p^* = 0.1446. \tag{29}$$

This result for p^* is in excellent agreement with the result 0.1453 ± 0.002 computed by Monte Carlo simulations (Jerauld *et al* 1984b). The percolation threshold, which is the so-called 'critical volume fraction', for a continuum percolation system was found to be $V_c = 0.15 \pm 0.01$ by Scher and Zallen (1970). Moreover, the critical percolation probability of a continuum percolation model proposed by Webman *et al* (1976) is $p_c = 0.145 \pm 0.05$ from the limiting percolation threshold of a percolation model.

The transformation has an eigenvalue $\lambda = 1.7211$. Thus the connectedness length exponent ν is given by

$$\nu = 1.2766. \tag{30}$$

The connectedness length exponent was found to be $\nu = 0.88 \pm 0.05$ for the Delaunay network from Monte Carlo simulations (Jerauld *et al* 1984b).

5. Discussion and conclusion

We have treated the various renormalisations for the critical behaviours of two- and three-dimensional percolating systems. The results for the eigenvalue λ, the fixed point p^*, and the connectedness length exponent ν of all the transformations are summarised in table 1.

A remarkable feature is that the non-trivial fixed point for the site problem on the Delaunay network in two dimensions is in complete agreement with the conjecture of

Table 1. Results of renormalisation group approach to Delaunay networks.

Number of elements		Scale		Connectedness	Fixed
Original	New	factor	Eigenvalue	length index	point
Two-dimensional bond lattice					
12	3	2	1.5929	1.4888	0.3213
Two-dimensional site lattice					
4	1	2	1.5	1.7095	1/2
9	1	3	1.9235	1.6794	1/2
9	4	1.5	1.2824	1.6303	1/2
Three-dimensional site lattice					
8	1	2	1.7211	1.2766	0.1446

Sykes and Essam (1964) for the critical percolation probability of the fully triangulated site lattice ($p_c = \frac{1}{2}$). The fact that the fixed point or the critical point is located at $\frac{1}{2}$ may be due to the self-matching property and the statistical isotropy (invariance under $\pi/2$ rotation) of the lattices in consideration. Self-matching occurs not only on a regular triangular lattice but also on any infinite planar lattice, all of whose faces are triangular. Therefore, any random Delaunay network in two dimensions is statistically self-matching. Furthermore the self-matching occurs on any 'finite' planar lattice with triangular faces.

Consider a finite planar lattice with linear size b which is statistically homogeneous; let $\mathcal{R}^{V}(p; b)$ be the probability that the finite lattice is vertically conductive and $\mathcal{R}^{H}(p; b)$ be the probability of its being horizontally conductive. When the lattice is statistically isotropic, i.e. when it has the same percolation properties for the vertical and horizontal directions, we have

$$\mathcal{R}^{V}(p; b) = \mathcal{R}^{H}(p; b) = \mathcal{R}(p; b). \tag{31}$$

In the site percolation process on self-matching lattices, the event that the system is vertically (horizontally) conductive is equivalent to the event that its matching system with exchange of present and absent sites is horizontally (vertically) non-conductive. Therefore it holds that

$$\mathcal{R}(p; b) + \mathcal{R}(1-p; b) = 1 \tag{32}$$

which is reduced for $p = \frac{1}{2}$ to a known expression

$$\mathcal{R}(\tfrac{1}{2}; b) = \tfrac{1}{2}. \tag{33}$$

This relation shows that our renormalisation group transformation gives the fixed points value $\frac{1}{2}$ for the two-dimensional Delaunay site network, provided that finite graphs chosen for renormalisation are statistically homogeneous, isotropic and simplicial.

As for critical percolation probabilities, the non-rigorous reasoning of Sykes and Essam leads to the conclusion that the critical probabilities for the square lattice bond model, the triangular lattice site model and the site model on any fully triangulated graph are all equal to $\frac{1}{2}$. Recently, Kesten (1980, 1982) and Russo (1981, 1982) have rigorously verified this conjectured value for the square lattice bond model and the triangular lattice site model. Furthermore, Kesten (1982) has proved that $p_c = \frac{1}{2}$ for the site model on a fully triangulated periodic planar graph with one symmetry axis.

However, counterexamples have been constructed by Van den Berg (1981) and Wierman (1984) against the conjecture that any fully triangulated graph has a common critical value $\frac{1}{2}$. Note that these counterexamples are neither homogeneous nor periodic.

In Kesten's argument it is demonstrated that three critical probabilities defined in different manners (the infinite cluster size critical probability, the mean cluster size critical probability and the sponge critical probability) are equal to one another. This implies that the limit of $\mathcal{R}(p; b)$ as $b \to \infty$ obeys a step function such that

$$\mathcal{R}(p; \infty) = \begin{cases} 0 & p < p_c \\ 1 & p > p_c \end{cases} \tag{34}$$

where p_c corresponds to the sponge critical probability. Also for statistically homogeneous, isotropic and simplicial graphs, we assume that the limit $\mathcal{R}(p; \infty)$ has the same form as (34) or simply assume that $\mathcal{R}(p; \infty)$ has only one point of discontinuity. According to (32), then, the point of discontinuity of $\mathcal{R}(p; \infty)$ should coincide with that of $1 - \mathcal{R}(1-p; \infty)$. Hence it follows immediately that

$$p_c = \tfrac{1}{2} \tag{35}$$

for any two-dimensional Delaunay network which is statistical homogeneous, isotropic and simplicial. Thus we conclude that in this case the fixed point of the renormalisation transformations and the critical percolation probability have the same value $\frac{1}{2}$.

In short, the percolation threshold of networks depends on the distribution of coordination numbers and the geometrical structure of tessellations. The bond threshold of the Delaunay network in two dimensions is slightly lower than that of the triangular lattice. The site percolation threshold of the Delaunay and triangular networks are exactly the same. In three dimensions, the site percolation threshold of the Delaunay networks are lower than that of all types of regular lattices.

The connectedness length exponent is independent of the lattice type both in the site and bond problems. This supports the hypothesis of super-universality that the exponent depends only on the dimensionality. Up to now, no value has been known for the connectedness length exponent for the Delaunay network in two dimensions. Estimates of ν for various regular lattices are found in the literature: $\nu = 1.35 \pm 0.03$ from a Monte Carlo simulation (Kertész *et al* 1982), $\nu = 1.333 \pm 0.002$ from the Potts model (Blöte *et al* 1981), $\nu = 1.33 \pm 0.07$ (Vicsek and Kertész 1981) and $\nu = 1.354 \pm 0.015$ (Reynolds *et al* 1978) from Monte Carlo renormalisation, and $\nu = 1.330$–1.332 on the site and bond lattices from the phenomenological renormalisation (Derrida and De Seze 1982). The estimates are in fairly good agreement with our results on the site and the bond Delaunay networks in view of the small size of cell. In three dimensions, the connectedness length exponent is calculated as $\nu = 0.88 \pm 0.05$ for the Delaunay network from Monte Carlo simulations (Jerauld *et al* 1984b). Heermann and Stauffer (1981) found $\nu = 0.89 \pm 0.01$ for regular lattices through Monte Carlo simulations. The best estimate of the exponent ν in three dimensions is $\nu = 0.88 \pm 0.02$ as found by Gaunt and Sykes (1983) using series expansions. The connectedness length exponents of the Delaunay network agree with those of regular networks obtained by our renormalisation procedure (Yuge and Murase 1978, Yuge 1978, 1979, Murase and Yuge 1979).

Although the above arguments are limited to the Delaunay network, we mention that the present procedure for employing the renormalisation group approach seems to be very promising for the study of other networks with the distribution of coordination numbers.

Appendix 1

$$R_1'(p') = p + p^2 - p^3$$

$$R_2'(p') = 2p - p^2$$

$$R_1^{(8)}(p) = 7p^2 - p^3 - 30p^4 + 41p^5 - 3p^6 - 35p^7 + 33p^8 - 13p^9 + 2p^{10}$$

$$R_1^{(7)}(p) = 5p^2 + 4p^3 - 24p^4 + 8p^5 + 34p^6 - 34p^7 - 3p^8 + 20p^9 - 11p^{10} + 2p^{11}$$

$$R_2^{(7)}(p)_x = 7p^2 - p^3 - 30p^4 + 41p^5 - 3p^6 - 35p^7 + 33p^8 - 13p^9 + 2p^{10}$$

$$R_2^{(7)}(p)_y = 6p^2 + 3p^3 - 35p^4 + 40p^5 + 6p^6 - 45p^7 + 38p^8 - 14p^9 + 2p^{10}$$

$$R_3^{(7)}(p) = 5p^2 + 6p^3 - 36p^4 + 34p^5 + 15p^6 - 50p^7 + 39p^8 - 14p^9 + 2p^{10}$$

$$R_1^{(6)}(p) = 4p^2 + 6p^3 - 21p^4 - 3p^5 + 42p^6 - 30p^7 - 12p^8 + 25p^9 - 12p^{10} + 2p^{11}$$

$$R_2^{(6)}(p) = 6p^2 + 3p^3 - 35p^4 + 40p^5 + 6p^6 - 45p^7 + 38p^8 - 14p^9 - 2p^{10}$$

$$R_3^{(6)}(p)_x = 6p^2 + 3p^3 - 33p^4 + 31p^5 + 22p^6 - 59p^7 + 44p^8 - 15p^9 + 2p^{10}$$

$$R_3^{(6)}(p)_y = 5p^2 + 6p^3 - 36p^4 + 34p^5 + 15p^6 - 50p^7 + 39p^8 - 14p^9 + 2p^{10}$$

$$R_4^{(6)}(p)_x = 6p^2 + 4p^3 - 34p^4 + 20p^5 + 58p^6 - 108p^7 + 79p^8 - 28p^9 + 4p^{10}$$

$$R_4^{(6)}(p)_y = 4p^2 + 6p^3 - 21p^4 - 3p^5 + 42p^6 - 30p^7 - 12p^8 + 25p^9 - 12p^{10} + 2p^{11}$$

$$R_1^{(5)}(p) = 5p^2 + 6p^3 - 29p^4 - p^5 + 87p^6 - 128p^7 + 86p^8 - 29p^9 + 4p^{10}$$

$$R_2^{(5)}(p) = 6p^2 + 3p^3 - 33p^4 + 31p^5 + 22p^6 - 59p^7 + 44p^8 - 15p^9 + 2p^{10}$$

$$R_3^{(5)}(p)_x = 5p^2 + 2p^3 - 14p^4 - 10p^5 + 44p^6 - 24p^7 - 21p^8 - 21p^9 + 30p^{10} + 2p^{11}$$

$$R_3^{(5)}(p)_y = 5p^2 + 6p^3 - 29p^4 - p^5 + 87p^6 - 128p^7 + 86p^8 - 29p^9 + 4p^{10}$$

$$R_1^{(4)}(p) = 6p^2 + 2p^3 - 22p^4 - 10p^5 + 98p^6 - 138p^7 + 91p^8 - 30p^9 + 4p^{10}.$$

Appendix 2

$$R_1^{(8)}(p) = 11p^3 - 20p^4 + 15p^5 - 7p^6 + 2p^7$$

$$R_1^{(7)}(p) = 9p^3 - 10p^4 - 3p^5 + 7p^6 - 20p^7$$

$$R_1^{(6)}(p) = 8p^3 - 6p^4 - 6p^5 + 12p^7 - 9p^8 + 2p^9$$

$$R_2^{(6)}(p)_x = 7p^3 - 3p^4 - 10p^5 + 5p^6 + 6p^7 - 5p^8 + p^9$$

$$R_2^{(6)}(p)_y = 9p^3 - 8p^4 - 8p^5 + 9p^6 + 2p^7 - 4p^8 + p^9$$

$$R_1^{(5)}(p) = 8p^3 - 6p^4 - 6p^5 + 12p^7 - 9p^8 + 2p^9$$

$$R_1^{(4)}(p) = 9p^3 - 11p^4 + 3p^5 - 7p^6 + 14p^7 - 9p^8 + 2p^9.$$

Appendix 3

$$R_1(p)_x = 9p^2 - 15p^3 + 5p^4 + 6p^5 - 5p^6 + p^7$$

$$R_1(p)_y = 9p^2 - 16p^3 + 9p^4 - p^6$$

$$R_1(p)_z = R_1(p)_x$$

$$R_2(p)_x = 9p^2 - 15p^3 + 5p^4 + 6p^5 - 5p^6 + p^7$$

$$R_2(p)_y = 9p^2 - 16p^3 + 9p^4 - p^6$$

$$R_2(p)_z = R_2(p)_x$$

$$R_3(p)_x = R_3(p)_y = R_3(p)_z = 9p^2 - 16p^3 + 9p^4 - p^6$$

$$R_4(p)_x = 9p^2 - 14p^3 + 16p^5 - 15p^6 + 6p^7 - p^8$$

$$R_4(p)_y = 9p^2 - 16p^3 + 9p^4 - p^6$$

$$R_4(p)_z = R_4(p)_x$$

$$R_5(p)_x = R_5(p)_y = R_5(p)_z = 9p^2 - 16p^3 + 9p^4 - p^6.$$

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